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The Neveu-Schwarz five-brane and its dual geometries

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Abstract: In this paper we discuss two aspects of duality transformations on the Neveu-Schwarz (NS) 5-brane solutions in type II and heterotic string theories. First we demonstrate that the non-extremal NS 5-brane background is U-dual to its CGHS limit, a two-dimensional black hole times $S^3 \times T^5$; an intermediate step is provided by the near horizon geometry which is given by the three-dimensional BTZ_3 black hole (being closely related to AdS_3) times $S^3 \times T^4$. In the second part of the paper we discuss the T-duality between k NS 5-branes and the Taub-NUT spaces respectively ALE spaces, which are related to the resolution of the A_{k-1} singularities of the non-compact orbifold $\mathbf{C}^2/\mathbf{Z}_k$. In particular in the framework of $N = 1$ supersymmetric gauge theories related to brane box constructions we give the metric dual to two sets of intersecting NS 5-branes. In this way we get a picture of a dual orbifold background $\mathbf{C}^3/\Gamma$ which is fibered together out of two $N = 2$ models ($\Gamma = \mathbf{Z}_k \times \mathbf{Z}_{k'}$). Finally we also discuss the intersection of NS 5-branes with D branes, which can serve as probes of the dual background spaces.

Keywords: p-branes, D-branes, Superstring Vacua, Black Holes in String Theory.

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Contents

1. Introduction

The NS 5-brane [1] is one of the first string soliton solutions, which can be constructed both for the type IIA/B superstrings as well as for the heterotic string. In certain limits there exists a CFT description of the NS 5-brane (see also [2]). In type IIB the NS 5-brane is S-dual to the D 5-brane [3].

The NS 5-brane plays an important role in the construction of gauge theories from branes [4]. In this context it serves as a kind of background for the D-branes on which the gauge fields live, since the NS 5-branes are heavy and the D-branes are light. It is known for some time that $k + 1$ parallel NS 5-branes are T-dual to the ALE space with A_k singularity [5], which is the local geometry of a K3 around a singularity. So now ALE is the background which is probed by D branes, which are then also called fractional branes [6]. If one considers intersecting NS 5-branes one gets a background for N=1 models (brane boxes) [7]. After T-duality one gets a space with a local $\mathbf{C}^3/\Gamma$, $\Gamma = \mathbf{Z}_k \times \mathbf{Z}_{k'}$ singularity. This describes the situation of a specific Calabi-Yau manifold around a singularity. The T-duality between the Hanany-Witten set up and the fractional branes was already recently discussed for $N = 2$ space-time supersymmetry in [8] and for the $N = 1$ brane box models in [9]. We will discuss several aspects of the duality among NS 5-branes and ALE.

Another aspect arises from the observation that D or M brane solutions are in fact U-dual to their own horizon geometry [10, 11]. This provides a relation to supergravity on anti-de Sitter spaces and, in the gauge theory picture, to a corresponding large N limit [12]. Via S-duality one expects an analogous behaviour for NS 5-branes. In fact we will explicitly demonstrate that the NS 5-brane background is U-dual to its CGHS limit via the intermediate step of BTZ_3 [13].

This paper is organized as follows. In the next section we present a short summary on the asymptotic geometry change of string background spaces using the heterotic [14] or respectively the type IIB [3] S-duality group. Then in section 3, applying the previous discussion, we show the equivalence of the non-extremal NS 5-brane string background to its own horizon geometry, namely the BTZ black hole respectively the CGHS limit [15], via U-duality. In section 4 we will turn the discussion to the T-duality between NS 5-branes and the ALE type of spaces. Here we especially focus on the construction of the dual background spaces for intersecting NS 5-branes versus six-dimensional non-compact orbifolds $\mathbf{C}^3/\Gamma$ which describe the local neighborhood around the singularities of a certain Calabi-Yau 3-fold.

2. Asymptotic geometry change

Recently it has been observed [10, 11] that via a sequence of duality transformations the metrics of M 2-, D 3- and M 5-branes with flat asymptotic geometry can be transformed to their own horizon geometries which correspond to the asymptotic non flat spaces $AdS_4 \times S^7$, $AdS_5 \times S^5$ and $AdS_7 \times S^4$, respectively. Here, one element within this series of duality transformations is a certain change, S, of coordinates (u, v) which has the form an $Sl(2, \mathbf{R})$ transformation:

$$S: \qquad \begin{pmatrix} v \\ u \end{pmatrix} \to \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}. \tag{2.1}$$

This transformation has the effect that it shifts to zero the constant part in the harmonic functions of the above p-brane metrics.

Another way to perform the described geometry change by removing the constant part in the metric goes back to the already older work of [16], where the asymptotically flat four-dimensional Taub-NUT space of the KK monopole was transformed into the non-flat ALE manifold via a TST duality transformation. In this context, S now means a genuine strong-weak coupling duality plus an axionic shift transformation which is again an element of $Sl(2, \mathbf{R})$. This approach was recently applied to D branes [17], in particular by dualizing D 3-branes to intermediates D instantons, and also to heterotic black holes in four and two dimensions [18].

In this section we like to apply analogous techniques to transform the metric of the NS 5-brane to its own horizon geometry via eliminating the constant part in the metric. For the extremal NS 5-brane this brings us to the socalled throat limit which can be described by an WZW type superconformal field theory. For the non-extremal NS 5-brane the sequence of duality transformations will bring us to the socalled CGHS limit [15].

First let us recall some facts about the strong-weak coupling S-duality group $SL(2, \mathbf{R})$ can be used to perform asymptotic geometry changes. The discussion goes in parallel both for the heterotic as well as type II NS 5-brane.

The two well-known frameworks for the non-perturbative group of S-duality symmetries are the 4-dimensional heterotic string on T^6 and the 10-dimensional type IIB string. The group $Sl(2)$ operates fractionally linear on the combination of axion and coupling constant e^{ϕ}, where in the heterotic case we have $S = a + ie^{-2\phi}$ with a being the Hodge dual of the four-dimensional $B_{\mu\nu}$-field; in the type IIB case S is given by $S = l + ie^{-\phi}$ with l being the RR scalar field.

We will see that for metrics flat in the Einstein frame the asymptotic geometry change (in the string frame) is caused, in the end, by the possible axion shift (in the hetrotic string) resp. RR scalar shift in the 7-brane picture (for the type IIB string), i.e. the well-known (F-theory) monodromy in the transversal complex plane (around the "singular elliptic fibre" at the 7-brane position, cf. the stringy cosmic string [19], and [20]).

The $Sl(2)$ S-duality group has the generators

$$\omega_1 := \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \qquad \omega_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The interpretation of these elements with respect to $S = x + iy$ is well-known:[1] ω_1 gives the shift[2] in x (axion for the heterotic string resp. RR-scalar for type IIB), ω_2 gives the coupling constant inversion $y \rightarrow -\frac{1}{y}$ (for vanishing x). One obvious question arises: besides the inversion element ω_2 and the upper triangular elements $\omega_1(B)$ one has the lower triangular elements which are conjugates of the upper triangular elements by the non-perturbative inversion element ω_2 (we write $\omega_1(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$):

$$\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} = \omega_2 \omega_1(-C) \omega_2^{-1}\,.$$

In the following we will elucidate their connection with the notion of *asymptotic geometry change* in both of the mentioned string theories, the heterotic string and the type IIB.

A suitably normalised element of the relevant subgroup is the shift

$$S_{\text{shift}} = \begin{pmatrix} 1 & 0 \\ -1/2 & 1 \end{pmatrix}. \tag{2.2}$$

[1]In our actual computations we are going to the Euclidean space where the complex S-field is replaced by $S_{\pm} = x \pm y$.

[2]Note that we have normalized x in such a way that x can be shifted by half integers.

For reasons explained in subsection 4.1 we will be especially interested in the case where $x = y$. In that case S_{shift} has the effect (regard footnote 4) of mapping again to a S' of $x' = y'$ where the effect on $y = e^{-j\phi}$ ($j = 2, 1$ for the heterotic resp. IIB case) can be described as

$$S_{\text{shift}} : y^{-1} \to y'^{-1} = y^{-1} - 1 \tag{2.3}$$

(and the same for x) which reflects the nature of the involved (upper triangular=ordinary) shift element conjugated by the inversion element.

In order to eventually perform the desired geometry change the S-duality transformation has to be combined with further elements of the U-duality group. For the heterotic string the additional elements are just T-duality transformations (see subsection 4.1 for the concrete treatment of T-duality), such that one ends up with the combined transformation of the form $TS_{\text{shift}}T$. On the other hand, for the type II geometry a more involved U-duality transformation is necessary, namely one has to consider the sequence $\mathcal{T}'S_{\text{shift}}\mathcal{T}'$ with $\mathcal{T}' := T_i S_{\text{inv}} T_{WV}$. Here T_i is a T-duality transformation along the direction i within the world volume of the 5-brane and T_{WV} denotes the T-dualisation of all world volume directions. The reason why one has to use a more involved sequence $\mathcal{T}'$ in the type II string compared to the simple T-duality in the heterotic case relies in the fact that for type II strings one is transforming the system to intermediate D instantons. Then one has a space-time interpretation for S_{shift}, acting like on the heterotic axion now on the type IIB RR-scalar field respectively, which amounts to an *asymptotic geometry change* by 'deleting the "1" in the harmonic function' (for the 4D heterotic space-time geometry resp. the 4D transversal directions to the 5-brane) (cf. [17]). Namely the background to which S_{shift} is applied to achieve this effect has to be **flat in the Einstein frame**, so that one has in the string frame

$$\begin{aligned} ds^2 &= V(\xi) d\xi^2 \\ e^{j\phi} &= V(\xi) \end{aligned} \tag{2.4}$$

As $y^{-1} = V$ the desired effect in the metric follows.

Let us describe briefly the process of asymptotic geometry change from the extremal type II NS 5-brane to its horizon geometry, i.e. to the socalled throat limit. As said before, the throat limit corresponds to 'deleting the "1" in the harmonic function' H_5 (this time in the transversal geometry of the 5-brane) and is achieved by the element of the U-duality group $\mathcal{T}'S_{shift}\mathcal{T}'$ where $\mathcal{T}' := TS_{inv}T_{WV}$ which changes the type IIA NS 5-brane to a type IIB D(-1) brane. Note that the (-1) brane carries the electric charge (measured by an integral over the S^9 of its transversal space, i.e. all of space-time) for the RR-scalar, whereas the Hodge-dual 7-brane carries the magnetic charge (measured by the l_{RR} upper triangular monodromy in $\mathbf{C}_{transversal}$).[3]

[3]Note that the physical interpretation of the formal U-duality process is not immediate (cf. [17, 12]) as the used T-dualities along (space-like) isometric directions which become light-like (null) at spatial infinity [11] are non-standard.

In the sequence $\mathcal{T}'$ the NS-NS B field is by S-duality of type IIB mapped to a RR B field the Hodge-dual 6-form of which becomes after T_{WV} the RR scalar. As the 5-brane metric is *flat in the Einstein frame*, which means essentially that $e^{2\phi} = H_5$, the shift in the inverse of $y = e^{-\phi} = H_5^{-1}$ leads to the desired effect in the metric (which in the string frame is just given by the harmonic function times the flat metric). So the metric is indeed dually mapped to some 'subsector' of itself, the near-horizon geometry.

So we see that, if we include the conjugating inversion element ω_2 of S_{shift} in the conjugation process $\mathcal{T}'$, that it is in the end really the (upper triangular) monodromy of the RR-scalar in the transversal complex plane of the dual 7-brane which causes the asymptotic geometry change. The 7-brane is magnetically charged for the RR-scalar (which is $\mathcal{T}'$-dual to the magnetic NSNS B-field of the NS 5-brane in type IIA we started with), which is detected by the mentioned monodromy related to the stringy cosmic string resp. the singular elliptic fibre of the associated F-theory situation.

3. U-duality of the NS five-brane with its CGHS limit

Now we want to describe in more detail the geometry change from the non-extremal type NS 5-brane to the socalled CGHS limit [15].

For this below a two step process is described which shows that the CGHS-limit,[4] $BH_2 \times S^3$, (which can be interpreted as an α' exact solution) of the near extremal NS 5-brane (cf. for example [21]),

$$NS_5 \longrightarrow BH_2 \times S^3_{\mathcal{Q}_5}\,, \tag{3.1}$$

is actually U-dual to it via the combination of the U-duality [11]

$$NS_5 \simeq BTZ_3 \times S^3 \times T^4 \tag{3.2}$$

with the T-duality [24]

$$BTZ_3 \times S^3 \times T^4 \simeq BH_2 \times S^3 \times (S^1 \times T^4)\,. \tag{3.3}$$

As this comes down to 'deleting the additive constant "1" in the harmonic function', i.e. a *change of asymptotic geometry*, this compares nicely with a corresponding duality (TST, the classical Ehlers-Geroch transform) between two purely gravitational backgrounds [16], which one also gets by 'deleting the additive constant "1" in the harmonic function', namely between the ALE instanton and the multi Taub-NUT space. This is a transformation operating purely in the 4-dimensional transversal space whereas the dualities shown above make use of transformations in the world-volume sector, too. Nevertheless the comparison matches nicely as the 5-brane is well known to be T-dual to the ALE-space [5] - but on a transversal direction of the 5-brane compactified on a circle (corresponding with the S^1_τ-fibration of the ALE-space), which on closer inspection has some quite non-trivial subtleties [25]. These issues are described in section 3.

[4] This limit is relevant for the new QFT's in D=6 and D=5 [22].

3.1. The CGHS limit NS 5 $\to BH_2 \times S^3$

For the near-extremal NS 5-brane in type IIA with its world-volume compactified on $S^1_1 \times T^4_{2345}$ (we suppress the flat spatial world-volume directions) one has [21]

$$\begin{aligned} ds^2 &= -(1-\frac{r_0^2}{r^2})dt^2 + (1+\frac{\mathcal{Q}_5\alpha'}{r^2})(\frac{dr^2}{1-\frac{r_0^2}{r^2}} + r^2 d\Omega_3^2)\,, \\ e^{2\phi} &= e^{2\phi_\infty}(1+\frac{\mathcal{Q}_5\alpha'}{r^2})\,, \end{aligned} \tag{3.4}$$

with $H = \mathcal{Q}_5\epsilon_3$ (cf. subsect. 3.2). Let us introduce the near-horizon coordinate σ with

$$r = r_0 \cosh\sigma\,, \tag{3.5}$$

the non-extremality-parameter α_5

$$\sqrt{\mathcal{Q}_5} = \frac{r_0}{\sqrt{\alpha'}}\sinh\alpha_5\,, \tag{3.6}$$

and the energy density parameter

$$\mu = \frac{r_0^2}{g^2\alpha'}\,, \tag{3.7}$$

(here $g := e^{\phi_\infty}$) which occurs [21] in the (string-frame) energy per unit 5-volume[5] $M_5 := \frac{M}{V_5} = \frac{1}{\alpha'^3(2\pi)^5}(\frac{\mathcal{Q}_5}{g^2}+\mu) = \frac{1}{\alpha'^3(2\pi)^5}\mu\cosh^2\alpha_5$. Then one gets

$$\begin{aligned} ds^2 &= -\tanh^2\sigma dt^2 + (\mu g^2\cosh^2\sigma + \mathcal{Q}_5)\alpha'(d\sigma^2 + d\Omega_3^2)\,, \\ e^{2\phi} &= g^2 + \frac{\mathcal{Q}_5}{\mu\cosh^2\sigma}\,. \end{aligned} \tag{3.8}$$

One sees that making the $g \to 0$ limit (CGHS-limit [15]), while keeping the energy density parameter μ at order one, corresponds to 'deleting the additive constant "1" in the harmonic function' $1+\frac{\mathcal{Q}_5\alpha'}{r^2}$; at the same time this causes the decoupling of the S^3 sector with $ds^2_{S^3} = \mathcal{Q}_5 d\Omega_3^2$ and $H = \mathcal{Q}_5\epsilon_3$, leading to the 2-dimensional black hole times the $SU(2)$ WZW model:

$$\begin{aligned} ds^2 &= -\tanh^2\sigma dt^2 + \mathcal{Q}_5\alpha' d\sigma^2, \\ e^{2\phi} &= \frac{\mathcal{Q}_5}{\mu\cosh^2\sigma}\,. \end{aligned} \tag{3.9}$$

For the sake of later comparison let us transform the coordinates back by $\bar{r} := \sqrt{\frac{\mu}{\mathcal{Q}_5}}\cosh\sigma = \bar{r_0}\cosh\sigma$:

$$\begin{aligned} ds^2 &= -(1-\frac{\bar{r}_0^2}{\bar{r}^2})dt^2 + \frac{\mathcal{Q}_5\alpha'}{\bar{r}^2}\frac{d\bar{r}^2}{1-\frac{\bar{r}_0^2}{\bar{r}^2}}\,, \\ e^{2\phi} &= \frac{1}{\bar{r}^2}\,. \end{aligned} \tag{3.10}$$

Let us remark that $g \to 0$ implied here that $r_0 \to 0$ as α' is kept fixed here. [12] makes $\alpha' \to 0$ instead of $g \to 0$, apart from that similar reduction to the near-horizon region.

[5] Here $V_5 = (2\pi)^5 R_1 R_2 R_3 R_4 R_5$.

3.2. The duality NS 5 $\simeq BTZ_3 \times S^3 \times T^4$

Let us start again with the NS 5-brane in type IIA with its world-volume compactified on $S^1_1 \times T^4_{2345}$ in the notation of [11] (set $H_1 = 1$ for now; let $dy^2 := dx_2^2 + \cdots + dx_5^2$):

$$\begin{aligned} ds^2 &= \frac{1}{H_1}[-(1-\frac{r_0^2}{r^2})dt^2 + dx_1^2] + dy^2 + (1+\frac{\mathcal{Q}_5\alpha'}{r^2})(\frac{dr^2}{1-\frac{r_0^2}{r^2}} + r^2 d\Omega_3^2)\,, \\ e^{2\phi} &= \frac{1}{H_1}(1+\frac{\mathcal{Q}_5\alpha'}{r^2})\,, \\ H_{ijk} &= \frac{1}{2}\epsilon_{ijkl}\partial_l(1+\coth\alpha_5\frac{\mathcal{Q}_5\alpha'}{r^2})\,. \end{aligned} \tag{3.11}$$

Note that ϕ is shifted by ϕ_∞ compared to the previous equations.

So we are in the case[6] $\mathcal{Q}_1 = \alpha_1 = 0$ of [11] where $H_i = 1 + \frac{\mathcal{Q}_i\alpha'}{r^2}$ and $\mathcal{Q}_i = \frac{r_0^2}{\alpha'}\sinh^2\alpha_i$ with $i = 1, 5$. Note that because of the $\coth\alpha_5 = \sqrt{1+\frac{r_0^2}{\mathcal{Q}_5\alpha'}}$ in H only for the extremal case of α_5 very large one has an axionic instanton. (Note that $g \to 0$ implies (by the μ condition) $r_0 \to 0$ and so $\alpha_5 \to \infty$.)

This is U-dual via $\mathcal{T}S_{\text{shift}}\mathcal{T}$ (with $\mathcal{T} := T_1 S T_{1234} S T_5$, cf. [11])[7] to a configuration with $H_1 = \frac{r_0^2}{r^2} = H_5$

$$ds^2 = \frac{r^2}{r_0^2}[-(1-\frac{r_0^2}{r^2})dt^2 + dx_1^2] + dy^2 + \frac{r_0^2}{r^2}(\frac{dr^2}{1-\frac{r_0^2}{r^2}} + r^2 d\Omega_3^2) \tag{3.12}$$

and $e^{2\phi} = 1, B_{01} = \frac{r^2}{r_0^2} - 1, H_{ijk} = \frac{1}{2}\epsilon_{ijkl}\partial_l(\frac{r_0^2}{r^2} - 1)$.

Note that the $\mathcal{Q}_5$-dependence, which seems to be lost, is still kept as the rescaling $R_5 \to R_5\cosh\alpha_5$ has happened. Because of the R_5-rescaling the 3-dimensional Newton constant is not[8] $\mathcal{G}_N^{(3)} = \frac{G_N^{(10)}}{V_{T^4}\cdot(r_0^3\Omega_3)}$ but $G_N^{(3)} = \frac{\mathcal{G}_N^{(3)}}{\cosh\alpha_5}$; so $G_N^{(3)}$ is here a function of r_0 and $\mathcal{Q}_5$.

Now by effectively 'deleting the additive constant "1" in the harmonic function' H_5 (besides changing $\mathcal{Q}_5\alpha'$ to r_0^2) the S^3 sector has decoupled where one has an S^3 of radius r_0 with $ds^2_{S^3} = r_0^2 d\Omega_3^2$ and $H = r_0^2\epsilon_3$. This leads to the structure $BTZ_3 \times S^3 \times T^4_y$ with the metric (besides $e^{2\phi} = 1, B_{t\varphi} = r_0(\frac{r^2}{r_0^2} - 1)$; $\varphi := x_1/r_0$)

$$ds^2_{BTZ} = -(\frac{r^2}{r_0^2} - 1)dt^2 + r^2 d\varphi^2 + \frac{dr^2}{\frac{r^2}{r_0^2} - 1}\,. \tag{3.13}$$

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[6] Also $\alpha_K = \mathcal{Q}_K = 0, H_K = 1$.

[7] We use here as a technical device the shift interpretation as coordinate change in the fundamental wave (reached from the IIB D-string, which we got from the IIA NS 5 brane after the part $T_1 S T_{1234}$ of $\mathcal{T}$, by the remaninig part ST_5); this is a technical alternative to the interpretation of the the wave in a 12D sense (the D-(-1) brane [28]) which one gets after part T_{05} which follows in the construction of $\mathcal{T}' = TST_{123405}$ described in the introduction. This gives the shift (deleting the "1") for H_5; it has actually to be coupled with a similar procedure for H_1 (cf. [11]). (Operator products to be read from the left.)

[8] Here $V_{T^4} = (2\pi)^4 R_2R_3R_4R_5$ is the volume in the beginning.

After the rescaling $t \to ct,\ \varphi \to c\varphi,\ r \to c^{-1}r,\ r_0 \to c^{-1}r_0 = \sqrt{Q_5\alpha'}$ of the metric by $c = \frac{r_0}{\sqrt{Q_5\alpha'}}$ it takes with $M_3 = c^2 = r_0^2/(Q_5\alpha')$ the form

$$ds^2 = -M_3(\frac{r^2}{r_0^2} - 1)dt^2 + r^2 d\varphi^2 + \frac{1}{M_3}\frac{dr^2}{\frac{r^2}{r_0^2} - 1}\,. \tag{3.14}$$

3.3. The non-compact untwisting $BTZ_3 \simeq BH_2 \times S^1$

To describe the 3-dimensional BTZ black hole [13] in its relation to anti-de Sitter space note first that AdS_3 is

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 = -l^2 \tag{3.15}$$

in the flat space of signature $(--++)$

$$ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2\,. \tag{3.16}$$

To get the physical coordinates for the black hole of mass M_3 with horizon at r_0 one makes first the coordinate change to r, φ and t (with $l^2 = \frac{r_0^2}{M_3}, \frac{r^2}{M_3} = x_1^2 - x_2^2, e^{\varphi\sqrt{M_3}} = \frac{\sqrt{M_3}}{r}(x_1 + x_2)$)

$$\begin{aligned}
x_0 &= \frac{r_0}{\sqrt{M_3}}\sqrt{1 - \frac{r^2}{r_0^2}}\cosh(t\frac{M_3}{r_0})\,,\\
x_1 &= \frac{r}{\sqrt{M_3}}\cosh\varphi\sqrt{M_3}\,,\\
x_2 &= \frac{r}{\sqrt{M_3}}\sinh\varphi\sqrt{M_3}\,,\\
x_3 &= \frac{r_0}{\sqrt{M_3}}\sqrt{1 - \frac{r^2}{r_0^2}}\sinh(t\frac{M_3}{r_0})\,,
\end{aligned} \tag{3.17}$$

and then identifies φ with period 2π to get BTZ_3 from AdS_3 [13, 24] leading to the metric (besides $e^{2\phi} = 1, B_{\varphi t} = \sqrt{M_3}r^2/r_0$)

$$ds^2 = -M_3(\frac{r^2}{r_0^2} - 1)dt^2 + \frac{1}{M_3}\frac{dr^2}{\frac{r^2}{r_0^2} - 1} + r^2 d\varphi^2\,. \tag{3.18}$$

If one makes a T-duality along the φ-direction, where one has a translational symmetry, one gets (after the further coordinate change[9] $\tilde{t} = \varphi/r_0, \tilde{\varphi} = \sqrt{M_3}t + \tilde{t}$) [24]

$$\begin{aligned}
\tilde{ds}^2 &= -(1 - \frac{r_0^2}{r^2})d\tilde{t}^2 + \frac{r_0^2/M_3}{r^2}\frac{dr^2}{1 - \frac{r_0^2}{r^2}} + d\tilde{\varphi}^2\,,\\
e^{2\phi} &= \frac{1}{r^2}\,,
\end{aligned} \tag{3.19}$$

[9] As φ is periodic, so are $\tilde{t}$ and $\tilde{\varphi}$; one actually works then on the covering space, to avoid CTC's.

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with $B_{\tilde{\varphi}\tilde{t}} = 0$. This is the 2-dimensional black hole times S^1. As AdS_3 is[10] (up to signature) $SL(2, \mathbf{R})$, and the 2-dimensional black hole is the $SL(2, \mathbf{R})$ WZW model with a $U(1)$ gauged [23], we see that the T-duality above is just a non-compact version of the T-duality-'untwisting' of $S^3 = SU(2)$ to $S^2 \times S^1 = SU(2)/U(1) \times U(1)$.

As $r_0^2/M_3 = \mathcal{Q}_5\alpha'$ we find coincidence with the crucial prefactor of the $d\bar{r}^2$ term in the CGHS limit.

4. Some aspects of the T-duality of the Taub-NUT spaces with the five-brane

We consider the T-duality for the ALE spaces with the five-brane. This was first made plausible by the argument of [5] that under a fibrewise T-duality for an elliptically fibered $K3$ with A_{k-1} singularity the monodromy for the complex structure parameter of the elliptic fibre (caused by the singularity) goes over to the monodromy for the Kähler parameter which gives the crucial H charge k. On closer inspection [25] this has some non-trivial points described below. (For a treatment from an other perspective cf. [26].)[11] The T-duality between ALE spaces and axionic instantons was also discussed in [27].

4.1. Case of one isometric direction

We will consider the case where we perform the T-duality with respect to one isometric direction. The spaces we are going to start with are the purely gravitational backgrounds given by the ALE and Taub-NUT spaces. Of these the ALE spaces describing the resolutions of the A_{k-1} singularities are complex two-dimensional non-compact relatives of $K3$, i.e. non-compact Ricci-flat hyperkaehler manifolds. The ALE manifold of the A_{k-1} series corresponds to the metric given by the Gibbons-Hawking multi-center ansatz

$$ds^2 = V(\vec{x})d\vec{x}^2 + V^{-1}(\vec{x})(d\tau + \vec{\omega} \cdot d\vec{x})^2 \,, \tag{4.1}$$

with the self-duality condition $\vec{\nabla} V = \vec{\nabla} \times \vec{\omega}$, where we are in the case $\epsilon = 0$ of

$$V = \epsilon + \sum_{i=1}^{k} \frac{1}{|\vec{x} - \vec{x_i}|} \,. \tag{4.2}$$

[10]The $\det g = 1$ condition for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbf{R})$ with $x_{0/3} = \frac{b \pm c}{2}, x_{2/1} = \frac{a \mp d}{2}$ translates to $l = i$ causing the signature change.

[11][26] argues that at a point in moduli space, where the center positions of the Taub-NUT metric (=KK monopole of IIa resp. M) merge (the critical point for gauge symmetry enhancement), and near the singularity (where the membranes wrapping the vanishing S^2 become massless giving the new non-abalian gauge bosons) the pol terms dominate so one can effectively neglect the "1" in the harmonic function leading to the ALE situation; then a TST transformation is made to the system of coalescing D6 branes in IIA (=KK monopole of M=Taub-NUT) where the mentioned membranes become the stretched strings between the D6 branes.

This space M_{k-1} is the smooth resolution of the singular variety $xy = z^k$ in $\mathbf{C}^3$ of type A_{k-1} with $\partial M_{k-1} = S^3/\mathbf{Z}_k$. The singular situation corresponds to the pol-terms coalescing: $V = \frac{k}{|\vec{x}|}$. The case $\epsilon = 1$ corresponds to the Taub-NUT spaces.

Now T-duality with respect to the $U(1)$-isometry generated by the Killing vector $\partial/\partial\tau$ gives with the well-known Buscher formula the conformal flat metric of the extremal NS 5-brane (cfr. eq. (3.4) with $r_0 = 0$)

$$ds^2 = V(\vec{y})(d\tau^2 + d\vec{y}^2)\,, \qquad B_{0i} = \omega_i\,, \qquad e^{2\phi} = V(\vec{y})\,, \tag{4.3}$$

where the self-duality condition for the original metric is now, in the new axion-dilaton sector, assuring the condition for an axionic instanton

$$H_{\mu\nu\rho} = \sqrt{g}\epsilon_{\mu\nu\rho}{}^{\sigma}\partial_\sigma\phi\,. \tag{4.4}$$

The H charge is from $H_{\mu\nu\rho} = \sqrt{g}\epsilon_{\mu\nu\rho}{}^{\sigma}\partial_\sigma\phi$ with $e^{2\phi} = V = \frac{k}{|\vec{x}|}$ easily seen to be $\frac{1}{2\pi^2}\int_{S^3} H = k$. This shows the appearance of the required H charge k.

The heterotic[12] axion a in the dual geometry is defined by dualizing the dual B field

$$\partial a = \pm e^{-2\phi} H_D^* = \frac{1}{2}V^{-2}\partial\omega^*\,. \tag{4.5}$$

The axion charge of the T-dual solution is called the nut charge of the original gravitational solution. One can say that the S-duality group is related to a duality between the electric aspects of the original gravity background (characterised by the Maxwell field $A = V^{-1}(d\tau + \vec{\omega}\cdot\vec{x}))$ and the magnetic aspects (characterised by the nut potential a). Note that the the isometry we are using is called 'translational' (the main importance of this is keeping the SUSY manifest after dualisation), i.e. the covariant derivative of the Killing vector field is self-dual which means $(\partial S_-)^2 = 0$. So from $\partial S_- = 0$ one gets $\partial V = \partial\omega^*$ so that V satisfies the 3D Laplace equation and one has $S_- = 0$, i.e. $a = V^{-1}$. We see that the Taub-NUT metric with can be mapped $S_\pm = a \pm e^{-2\phi} = a \pm V^{-1}$ to the ALE space via the shift $V \to V - 1$.

Note however that in $e^{2\phi} = \frac{k}{|\vec{y}|}$ the 3-dimensional harmonic function $\frac{1}{r_3}$ occurs whereas in the five-brane the 4-dimensional harmonic function $\frac{1}{r_4^2}$ occurs. The reason is of course that by doing T-duality in the periodic τ-direction in Taub-NUT space one arrives at the five-brane with one of its four transverse directions compactified on a circle. In other words, since the harmonic function of the original extremal 5-brane metric also depends on τ, before the T-duality from the 5-brane to the ALE space one has to *enforce* an isometric direction by taking the transversal space to be $R^3_{\vec{x}} \times S^1_\tau$ and requiring H_5 to be a 3D harmonic function $H_5 = V = 1 + \frac{Q_5}{r_3}$ (here $r_3 := |\vec{x}|$) in R^3 and independent of the S^1_τ direction. More precisely if the original 5-brane metric is scaled so that the τ-direction is very large and the $\vec{x}$-space looks correspondingly contracted,

[12] A analogous argument can be made for the type II RR-scalar after dualizing down to the D-instantons.

then in the dual space the direction of the corresponding little circle is 'suppressed', i.e. the ansatz there leads to a harmonic function in only three variables.

But note that this is only *one* possibility to realise the duality. One could also tune directly the merging without making the radius R of the S^1_τ large; then the dual circle (of radius $\tilde{R} = \alpha'/R$) is not 'suppressed'.

More concretely one has in the dual picture the Fourier decomposition along the dual circle [25]

$$e^{2\phi}(\vec{x}, \tau) = \sum_{n\in\mathbf{Z}} e^{in\tau/\tilde{R}}\Psi_n(\vec{x})\,, \tag{4.6}$$

where (the Ψ_n are no longer suppressed for $\tilde{R}$ being no longer small)

$$\Psi_0 = e^{2\phi_0} + \frac{k\alpha'}{2r\tilde{R}}\,, \qquad \Psi_n = \frac{\alpha'}{2r\tilde{R}}e^{-|n|r/\tilde{R}}e^{-in\tau_0/\tilde{R}} \qquad (n \neq 0)\,. \tag{4.7}$$

One can see that the Ψ_n interpolate between the 3-dimensional and the 4-dimensional harmonic function according to $\tilde{R}$ being very small or very large [29]. Furthermore the occurence of these momentum modes in the dual picture leads to the idea that in the original picture winding modes have to be included in the description.

This can also be understood from the following perspective. In the original picture of the ALE space one has besides the degree of freedom $\vec{x}_i$ (positions of the centers) also to take into account the parameters $\int_{\Sigma_i} B$ (the Σ_i being the non-trivial 2-cycles 'between the centers');[13] that they play an important role in the game was the insight of [30], who showed that actual gauge symmetry enhancement occurs only, if not only the positions of the centers merge (giving the A_{k-1} singularity), but also the B-field parameters have the value zero (and not the CFT orbifold value π; this leads to the breakdown of the CFT reasoning, necessary for the non-perturbative gauge symmetry enhancement). Now each of these 4 real parameters (for $i = 1\cdots, k-1$), the center-distance and the B-field parameter, constitute a hypermultiplet. In the dual picture this corresponds to the positions of the 5-branes in the transversal space $R^3\times S^1$. But the position parameter in the (dual) S^1-drection breaks the expected isometry in this direction.

This leads to the alternative view on the necessity of including winding modes in the original Taub-NUT picture. Above we saw this was caused by the actual 'occurence' (being no longer suppressed) of the S^1 in the 5-brane picture which lead to the momentum modes there and so to the winding modes in the original picture. Here we see that the S^1-position degree of freedom in the 5-brane picture corresponds in the original picture to the $\int_{\Sigma_i} B$ degree of freedom; but one gets the necessary compact S^2-cycles Σ_i 'between the centers' exactly beacuse the S^1-fibration in the τ variable in the Taub-NUT space collapses to circles of zero radius at the center points; and this

[13] Being given by the S^1_τ-fibration over the line in $R^3_{\vec{x}}$ connecting two centers; as the S^1 shrinks at the centers this is an S^2.

means that the corresponding winding modes there become light and so winding modes should be included in the description.

At the end of this section we briefly give the duality transformation for the non-extremal 5-brane. Its metric becomes

$$ds^2 = -(1-\frac{r_0}{r_3})dt^2 + H_5(\frac{dr_3^2}{1-\frac{r_0}{r_3}} + r_3^2 d\Omega_2^2 + d\tau^2)\,, \tag{4.8}$$

with the H-monopole magnetic field $H_{\tau\theta\varphi} = -\partial_\theta B_{\tau\varphi} = r_3^2 \sin\theta \partial_{r_3}(1+\coth\alpha_5 \frac{\mathcal{Q}_5}{r_3})$.

Then the non-extremal KK-monopole dual to non-extremal 5-brane is

$$ds^2 = -(1-\frac{r_0}{r_3})dt^2 + V^{-1}(d\tau + \vec{\omega}\cdot d\vec{x})^2 + V(\frac{dr_3^2}{1-\frac{r_0}{r_3}} + r_3^2 d\Omega_2^2)\,, \tag{4.9}$$

with KK magnetic field $F = \partial\vec{\omega}$. More precisely if $\vec{\omega} = A_\varphi$ then $F_{\theta\varphi} = -\partial_\theta A_\varphi = r_3^2 \sin\theta \partial_{r_3}(1+\coth\alpha_5 \frac{\mathcal{Q}_5}{r_3})$ with $\mathcal{Q}_5 = r_0 \sinh^2\alpha_5$.

In the case of two isometries one will see an effective reduction by two dimensions down to a function harmonic in two dimensions, the logarithm. As there is an effective 2+2 split of the coordinates and in view of the hyper-Kähler nature of the relevant background, it is appropriate to describe the situation in an complex superfield formalism, the general features of which we describe first (cf. for ex. [31]).

4.2. Superfield formalism of Buscher duality and two isometries

The $N=2$ superspace action for one chiral ($\bar{D}_\pm U = 0$) superfield U and one twisted chiral ($\bar{D}_+ V = D_- V = 0$) superfield V is determined by the real potential function $K(U,\bar{U},V,\bar{V})$

$$S = \frac{1}{2\pi\alpha'}\int d^2x D_+ D_- \bar{D}_+ \bar{D}_- K(U,\bar{U},V,\bar{V})\,,$$

with the target space interpretation ($K_u := \frac{\partial K}{\partial u}$)

$$S_{bos} = -\frac{1}{2\pi\alpha'}\int d^2x (K_{u\bar{u}}\partial^a u \partial_a \bar{u} - K_{v\bar{v}}\partial^a v \partial_a \bar{v} + \epsilon_{ab}(K_{u\bar{v}}\partial^a u \partial^b \bar{v} + K_{v\bar{u}}\partial^a v \partial^b \bar{u}))\,,$$

which shows the $G_{\mu\nu}$ and the $B_{\mu\nu}$ part; so, for example, the H field components become

$$\begin{aligned} H_{u\bar{u}v} &= K_{u\bar{u}v}\,, \quad H_{v\bar{v}u} = K_{v\bar{v}u} \\ H_{u\bar{u}\bar{v}} &= -K_{u\bar{u}\bar{v}}\,, H_{v\bar{v}\bar{u}} = -K_{v\bar{v}\bar{u}}\,. \end{aligned} \tag{4.10}$$

Furthermore the string equations of motion have to be satisfied (vanishing β-function equations). If the central charge deficit (determined by the dilaton β-function) vanishes, one actually has $N=4$ supersymmetry in two dimensions.

In general one gets $N=4$ supersymmetry for a potential K satisfying the Laplace equation $K_{u\bar{u}} + K_{v\bar{v}} = 0$ (this is the generalization of the hyper-Kähler condition for backgrounds including a B field; it is only a sufficient condition in case of non-trivial dilaton). From the string equations of motion one has then

$$\begin{aligned}\partial_u \log K_{v\bar{v}} &= 2\partial_u \phi \\ \partial_v \log K_{u\bar{u}} &= 2\partial_v \phi\,,\end{aligned} \tag{4.11}$$

giving $e^{2\phi} \sim K_{u\bar{u}}$ so that the metric is flat in the Einstein metric $G^{\rm Einst}_{\mu\nu} = e^{-2\phi} G^{\sigma}_{\mu\nu}$, i.e. only the axion-dilaton sector is non-trivial and one has

$$H_{u\bar{u}v} = K_{u\bar{u}v} = 2e^{2\phi}\partial_v\phi \tag{4.12}$$

and $d\phi = \pm\frac{1}{2}e^{-2\phi}H^*$, the self-duality condition for the axion-dilaton sector.

For T-duality one has to assume the existence of (at least) one $U(1)$-isometry; this cooresponds to a Killing symmetry of the potential K

$$K = K(u+\bar{u}, v, \bar{v})\,.$$

The duality will trade in a twisted field w for the untwisted field u by a Legendre transformation leading to the dual potential ($r := u+\bar{u}$) [32], [31]

$$\tilde{K}(r, w+\bar{w}, v, \bar{v}) = K(u+\bar{u}, v, \bar{v}) - r(w+\bar{w})$$

i.e. after the variation w.r.t. u: $\frac{\delta S}{\delta u} = 0 \Rightarrow w + \bar{w} = K_r = K_u$ the independent variables for $\tilde{K}$ are $w, \bar{w}, v, \bar{v}$. As these are now only twisted fields (a set containig only untwisted fields would of course do it equally well) $\tilde{K}$ is a true Kähler potential providing a Kähler metric with Ricci-tensor

$$\tilde{R}_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log\det\tilde{G}_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log -\frac{K_{v\bar{v}}}{K_{u\bar{u}}}\,,$$

i.e. the dual background is Ricci-flat for $K_{v\bar{v}} \sim K_{u\bar{u}}$.

In the case of 2 translational $U(1)$ Killing symmetries

$$K = K(u+\bar{u}, v+\bar{v})$$

the Laplace equation is solved by ($u := r_1 + i\theta, v := r_2 + i\phi, z := r_1 + ir_2$)

$$K(r_1, r_2) = iT(r_1+ir_2) - i\bar{T}(r_1 - ir_2) = -2{\rm Im}T(z)\,,$$

where $T(z)$ is an arbitrary holomorphic function and the associated metric is

$$ds^2 = -4{\rm Im}T_{zz}(dud\bar{u} + dvd\bar{v})\,. \tag{4.13}$$

For our axionic instanton background consisting of the 5-brane with two isometric directions the relevant harmonic function is now just $H_5 = k\log|z|$.

Then the dual metric becomes (with $w = \frac{1}{2}K_u + i\theta$) [31]

$$\begin{aligned} ds^2 &= \frac{1}{K_{u\bar{u}}}(dw - K_{uv}dv)(d\bar{w} - K_{u\bar{v}}d\bar{v}) - K_{v\bar{v}}dvd\bar{v} \\ &= \mathrm{Im}S dz d\bar{z} + \frac{1}{\mathrm{Im}S}(d\theta - Sd\phi)(d\theta - \bar{S}d\phi)\,, \end{aligned} \tag{4.14}$$

with $S(z) := -\frac{1}{2}T_{zz}(z)$.

We interpret this as a part, local in the base (z-variable), of an elliptic fibration (cf. the discussion of the stringy cosmic string [19]), which is thus dual to the axionic instanton background we started with. In [19] also global issues in the base were treated making S a true well-defined modular invariant by multyplying it by an η-function term, i.e. $S(z)$ is just a local version of $\tau(z)$. If one specialises to an A_{k-1} singularity one has k cosmic strings at $z = 0$, i.e. $j(\tau(z)) = \frac{1}{z^k}$. Now at $\tau \approx i\infty, j \approx \infty$ one has $j(\tau) \sim e^{-2\pi\tau(z)}$ or $\tau(z) = -\frac{k}{2\pi}i \log z$, so

$$\mathrm{Im}S(z) = -\frac{k}{2\pi}\log|z|\,. \tag{4.15}$$

Note that the H charge of the original axionic instanton background is from $H_{u\bar{u}v} = K_{u\bar{u}v} = 2e^{2\phi}\partial_v\phi$ found to be proportional to n as the dilaton was $e^{2\phi} \sim K_{u\bar{u}} = -2\mathrm{Im}T_{zz} = 4\mathrm{ImS}$, i.e.

$$e^{2\phi} \sim k\log|z|\,, \tag{4.16}$$

which shows the consistency of the interpretation.

Note that the ALE description (which is local around the (resolved) singular point) is related to the resolution of the singularity, whereas the description given here in the fibration picture (which is local in the base around the (desingularised) fibre, but global in the fibre) is related to the deformation of the singularity to an elliptic fibration of smooth total space.

4.3. Intersection of two NS 5-branes – $N = 1$ brane boxes

In the gauge-theory-from-branes setup the NS 5-branes are considered to be heavy relative to the D4-branes whose world-volume gives the gauge theory. So the NS 5-branes (later, after T-duality, the KK-monopoles, resp., if one is interested in the singular situation at the neighborhood of the singularity, the ALE spaces in the transversal dimensions) constitute the 'background', the D4-branes (later, after T-duality, the fractional D3-branes) are the 'probes'.

So there are really two levels of consideration here: first the gauge theory where gravity is turned off and the light D3-branes; second there is a background, probed by the D3-branes, of ALE/KK-monopoles, the T-dual of the NS 5-branes which are considered to be heavy. In the case of $N = 2$ space-time supersymmetry with parallel NS 5-branes the background is just the well-known ALE/KK-monopole space; in the

$N = 1$ case it is a background of two ALE/KK-monopole spaces fibered together over a common R^2 direction. This back ground arises as the T-dual of intersecting NS and NS' 5-branes which build the socalled brane boxes of [7], as we will discuss in the following.

If on the other hand the backreaction of the D3-branes on the background is included, the former NS 5-branes (resp. their T-duals) become dynamical and it is appropriate to give a common metric for the total brane system. This will be the topic of the next subsection.

Let us describe the metric for the $N = 1$ situation of a $\mathbf{C}^3/\Gamma$ singularity, $\Gamma = \mathbf{Z}_k \times \mathbf{Z}_{k'}$, (probed by the D3-branes) [9] in the case of "adding up" two ALE spaces. Let us forget about the D-branes and just concentrate on the two now non-parallel NS 5-branes. Specifically to compute the metric of the NS-NS' 5-brane system one starts one step earlier with k D5-branes in 012345 and k' D5-branes in 012367 with compact directions 4 and 6. This has the metric [33] (with $e^{-2\phi} = H_5 H_{5'}$)

$$ds^2 = \frac{1}{\sqrt{H_5 H_{5'}}} ds^2_{0123} + \sqrt{\frac{H_{5'}}{H_5}} ds^2_{45} + \sqrt{\frac{H_5}{H_{5'}}} ds^2_{67} + \sqrt{H_5 H_{5'}} ds^2_{89}\,. \tag{4.17}$$

Then this is S-dualised to k NS 5-branes in 012345 and k' NS' 5-branes in 012367 giving

$$ds^2 = ds^2_{0123} + H_{5'} ds^2_{45} + H_5 ds^2_{67} + H_5 H_{5'} ds^2_{89}\,, \tag{4.18}$$

with $e^{2\phi} = H_5 H_{5'}$.

Then one makes T-dualities in the compact directions 4 and 6 giving (at the singularity) an A_{k-1} in 6789 and an $A_{k'-1}$ in 4589.

This "adding up" of the two ALE spaces is difficult to perform in the usual representationfor the ALE metric which has a 3+1 split in the coordinates. Instead one would like to have a representation which isolates the singularity in a $2 + 2_{89}$ description. This is provided by the stringy cosmic string description of subsection 4.2 which gives the A_{k-1} singularity in an elliptic fibration over $\mathbf{C} = R_{89}$ (so here the directions 5 and 7 are considered to be compact too; the respective "τ-circles" of the ALE spaces A_{k-1} in 6789 and an $A_{k'-1}$ in 4589 are well known to shrink to zero radius at the, in the merging limit common, center; they are the vanishing S^1's in the elliptic fibration).

So one assumes that H_5 (and correspondingly for $H_{5'}$), which — to make T_6 — was assumed to be independent of x_6 and just living as a harmonic function in 789, is now actually independent also of x_7 and so lives just as a logarithmic function in 89 (cf. eq. (4.16) above: $e^{2\phi} \sim k \log |z_{89}|$).

Because of the fibration structure now this type of representation of the ALE metric is - in contrast to the 3+1 representation - easily "added up". So this extends the $N = 2$ supersymmetric case with the (local in the base) description of an A_{k-1} singularity of an elliptic fibration (over $\mathbf{C}_{89}$) to the $N = 1$ supersymmetric case of a description[14] of the singularities of the doubly elliptically fibered Calabi-Yau space[15] $CY^{19,19} =$

[14] G. C. thanks A. Uranga for discussion on this point

[15] $dP_9 = \frac{1}{2}K3$ is the elliptically fibered surface $dP_9 = \left[\begin{smallmatrix} P^2 \\ P^1 \end{smallmatrix}\middle|\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$.

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$\begin{bmatrix} P^2_{z_{45}} & 3 & 0 \\ P^1_{z_{89}} & 1 & 1 \\ P^2_{z_{67}} & 0 & 3 \end{bmatrix} = dP_9 \times_{P^1} dP_9$. This Calabi-Yau space can be constructed as a $T^6/\mathbf{Z}_k \times \mathbf{Z}_{k'}$ orbifold, where we have identified $z_{45} = x_4 + ix_5,\ z_{67} = x_6 + ix_7,\ z_{89} = x_8 + ix_9$. Locally around the singularities we therefore consider the non-compact space $\mathbf{C}^3/\mathbf{Z}_k \times \mathbf{Z}_{k'}$, where the $\mathbf{Z}_k \times \mathbf{Z}_{k'}$ orbifold action, giving a genuine $\Gamma \subset SU(3)$, is fibered together by corresponding actions giving an A_{k-1} resp. $A_{k'-1}$ singularity in 6789 resp. 4589. Concretely the corresponding total metric has the following explicit form:

$$\begin{aligned} ds^2 = {} & ds^2_{0123} + \frac{1}{\mathrm{Im}S_{k'}(z_{89})}(d\theta_4 - S_{k'}(z_{89})d\phi_5)(d\theta_4 - \bar{S}_{k'}(\bar{z}_{89})d\phi_5) + \\ & + \frac{1}{\mathrm{Im}S_k(z_{89})}(d\theta_6 - S_k(z_{89})d\phi_7)(d\theta_6 - \bar{S}_k(\bar{z}_{89})d\phi_7) + \\ & + \mathrm{Im}S_k(z_{89})\mathrm{Im}S_{k'}(z_{89})dz_{89}d\bar{z}_{89}\,. \end{aligned} \tag{4.19}$$

Here $S_k(z_{89}) = \frac{k}{2\pi i}\log z_{89}$ and analogously for $S_{k'}$.

This metric descibes a fibration of a Γ-singularity in the z_{4567} respectively z_{67} directions over a singular point in the common base space with coordiates z_{89}, just like the $CY^{19,19}$ is a $T^2 \times T^2$ fibration over the common base P^1. The A_{k-1} singular point of one dP_9 direction times the S^1 of the remaining z-plane gives the complex curve of singularities $z_{67} = z_{89} = 0$ resp. $z_{45} = z_{89} = 0$ intersecting the S^5 relevant to the $AdS_5 \times S^5/(\mathbf{Z}_k \times \mathbf{Z}_{k'})$ in an S^1 of singularities given by the unit circle in z_{45} resp. z_{67} (cf. [9]; if k and k' are not coprime there exists a third curve of singularities).

4.4. Intersection of D 4-brane with NS 5-branes

Finally we want to describe the metric for the configuration of n D4-branes between k parallel NS 5-branes in type IIA, corresponding to the $N = 2$ supersymmetric $SU(n)^{k-1}$ gauge theory [34]. Let us start two steps earlier with D5-branes in type IIB, from which we get the NS 5-branes by type IIB S-duality, and D3-branes, being invariant under the S-duality, from which we get the D4-branes in type IIA by T-duality.

Starting with the intersection of D5-branes with D3-branes we obtain after a S-duality transformation the following metric which describes the intersection of NS 5-branes with D3-branes ($e^{2\phi} = H_5$):

$$\text{NS5} \perp \text{D3}: \quad ds^2 = \frac{1}{\sqrt{H_3}}ds^2_{012} + \sqrt{H_3}s^2_{345} + \frac{H_5}{\sqrt{H_3}}ds^2_6 + H_5\sqrt{H_3}ds^2_{789}\,. \tag{4.20}$$

Next let us perform a T-duality transformation with respect to the x_3 direction. Under this duality transformation the type IIB D3-brane turns into a type IIA D4-brane, i.e. one of the transverse directions of the D3-brane becomes a worldvolume direction of the D4-brane. Here we have to assume that H_3 is independent of x_3 and following [33] we may write $\tilde{H}_3 = H_4$. Finally let us apply a T-duality with respect to x_6 which leads to a multi Taub-NUT configuration and a fractional D3-brane (where $\omega_i = B_{6i}$

for $i = 7, 8, 9$ and $e^{2\phi} = 1$)

$$ds^2 = \frac{1}{\sqrt{H_3}} ds^2_{0123} + \sqrt{H_3}[ds^2_{45} + \frac{1}{H_5}(dx_6^2 + \vec{\omega} d\vec{x}_{789})^2 + H_5 ds^2_{789}]\,. \tag{4.21}$$

Inspection of this metric clearly exhibits the D3-brane with world volume along the (0123)-directions as well as the Taub-NUT space in the transversal directions (6789). Now one can proceed as before and 'delete' the constant in the harmonic function H_5 via a U-duality transformation. In this way the D3-brane is localized at the A_{k-1} singularity in the transversal space (6789). In addition one can also consider the limit where the constant part in the harmonic function H_3 can be neglected. In this case the geometry becomes equivalent to $AdS_5 \times S^5/\mathbf{Z}_k$. This space describes $N = 2$ supersymmetric gauge theories in the large N-limit, where the theories are supposed to be superconformal. Finally let us remark that we can also consider the combined system of D-branes which are positioned in the brane boxes of intersecting NS 5-branes, as described in subsection 4.3. The dual geometry of this set up is then given by D3-branes plus a Γ singularity, which extends into the full transversal space with directions (456789). At the horizon of the D3-branes the large N-limit of $N = 1$ supersymmetric gauge theories, based on the space $AdS_5 \times S^5/(\mathbf{Z}_k \times \mathbf{Z}_{k'})$, is obtained.

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